Hybrid finite-volume/finite-element simulations of fully-nonlinear/weakly dispersive wave propagation, breaking, and runup on unstructured grids

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- D. Lannes (IMB Bordeaux, France), F. Marche (U. Montpellier, France): Boussinesq modelling
- R. Pedreros & S. LeRoy (BRGM, France), R. Ata (EDF Chatou, France): Tohoku tsunami data

Some references

Source material

1. MR and A.G. Filippini, J.Comput.Phys. 271, 2014
2. A.G. Filippini, M. Kazolea and MR, J.Comput.Phys. 310, 2016
3. A.G. Filippini, M. Kazolea and MR, ISOPE proc.s , 2017

see also the PhD of A.G. Filippini (December 2016, available online on hal)

Additional references/related work

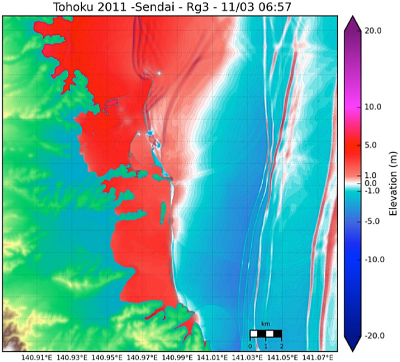
1. J.T. Kirby and G. Wei, J. Waterway, Port, Coastal, and Ocean Eng. 1995;
2. F. Shi, et al. Ocean Modelling, 2012
3. M. Kazolea et al., Coast.Eng. 2012 and J.Comput.Phys, 2014
4. N. Aissiouene et al, Networks and Heterogeneous media, 2016

Tohoku example

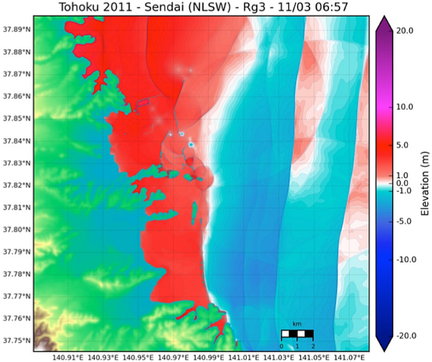
Tohoku example

Wave arrival in Sendai Bay

Tohoku example



Boussinesq



Shallow Water

Modelling approach 1/6

Near shore: wave transformation due to interaction with complex bathymetries

- strong nonlinearity and dispersion ;
- refraction and diffraction ;
- shoaling;
- breaking;
- induced currents;
- run-up and inundation;

2004 Sumatra tsunami reaching the coast of Thailand

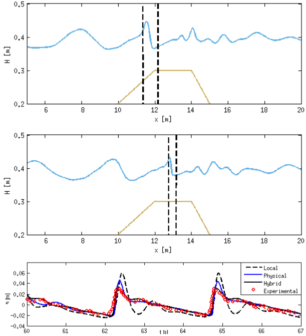
Sea waves diffracting around a peninsula

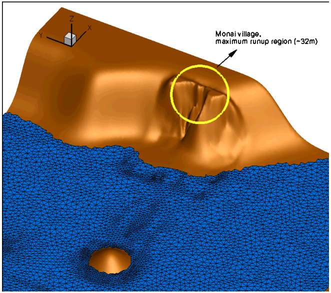
Rip current, Park beach (Coffs harbour, NSW Australia)

Modelling approach 2/6

Near shore hydrodynamics: modelling standpoint

(Ribbed channel clip)





Propagation: large scales, dispersion, shoaling, etc

POTENTIAL FLOW

Wave vreaking: dissipation, vorticity

CLOSURE MODEL

Runup/flooding : hydrostatic shallow water

SHALLOW WATER

Modelling approach 4/6

Near shore hydrodynamics: modelling standpoint

Propagation *vs* wave breaking closure

1. Potential/dispersive PDE for propagation + 3D Navier-Stokes (or SPH) for breaking/impact ;
2. Dispersive PDE for propagation + eddy viscosity to model dissipation in surf zone/breakers ;
3. Coupling dispersive PDEs with shallow water/hydrostatic limit:
   - Kirby, Grilli, et al (FUNWAVE-TVD)
   - Lynett et al USC (COULWave)
   - Delis, Kazolea, Synolakis (TUCWave) Coast.Eng. 2011, JCP 2014
   - Smit, Zijlema et al DELFT (SWASH)
   - See also:
     Tonelli, Petti Coast.Eng. 2009, Bonneton et al. JCP 2011, Coast.Eng. 2012

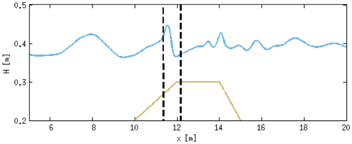
Modelling approach 5/6

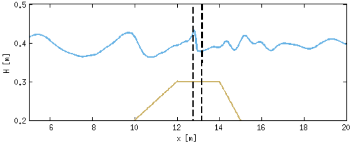
Breaking closure: shallow water dissipation

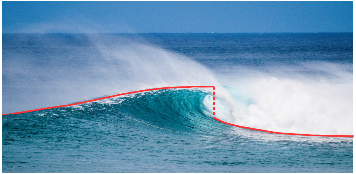
Closure model

1. Detect breaking regions
2. Remove dispersive terms
3. $\rightarrow$ shallow water shock
4. Total energy $E = gh^2/2 + hu^2/2$
5. Dissipation :

$$\mathcal{D}_b = [\mathcal{F}_E - \sigma E] \approx \gamma_b [H]^3$$







Modelling approach 5/6

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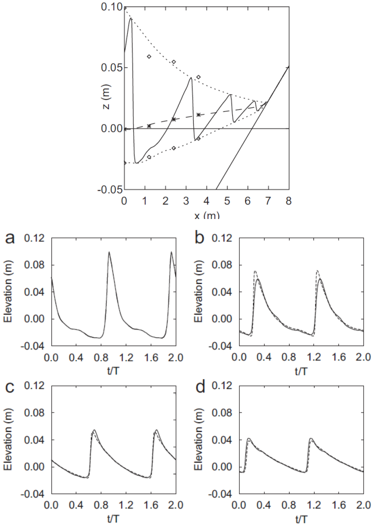

Modelling approach 6/6

Breaking closure: shallow water dissipation

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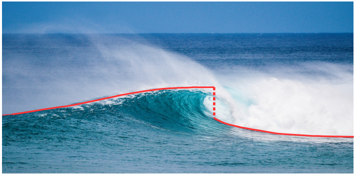


Need to handle shallow water:

- "Upwinding"/numerical dissipation
- Shock capturing in breaking regions
- etc

All the std. artillery....

Dispersive models for propagation

Which dispersion dominates ?

1. Continuous dispersive models have a range of validity related to the **model dispersion error**
2. Discrete dispersive models have a range of validity related to the **scheme dispersion error**

How can we exploit this knowledge to construct

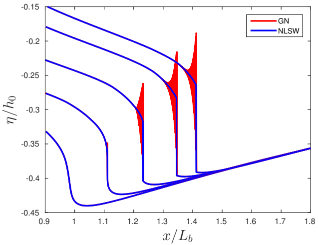
efficient low dispersion schemes on unstructured grids ?

Dispersive surface waves: Boussinesq approach (1/8)

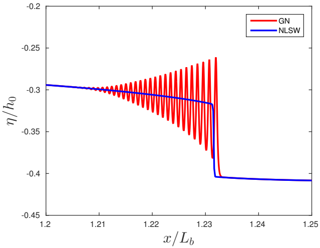
In fluid dynamics, **dispersion** of water **waves** generally refers to frequency **dispersion**, which means that **waves** of different wavelengths travel at different phase speeds. Water **waves**, in this context, are **waves** propagating on the water surface, with gravity and surface tension as the restoring forces.

Dispersion (water waves) - Wikipedia
https://en.wikipedia.org/wiki/Dispersion_(water_waves)

Undular bore (Garonne river, Bonneton et al *J.Geophys.Res.* 2015)



Tidal wave distorsion in a converging estuary

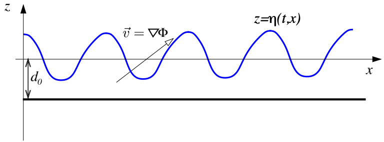


Dispersive surface waves: Boussinesq approach (2/8)

Linear travelling waves: incompressible Euler equations

Theory due to (G.B. Airy, *Encyclopædia Metropolitana*, 1841)

$\eta = A\,\cos(\kappa x - \omega t)$	travelling wave
$\Phi = B(z)\,A\,\sin(\kappa x - \omega t)$	potential
κ	wave number
$C = \omega/\kappa$	phase velocity



$$B(z) = \frac{g}{\omega}\,\frac{\cosh(\kappa(d_0 + z))}{\cosh(\kappa(d_0)}$$

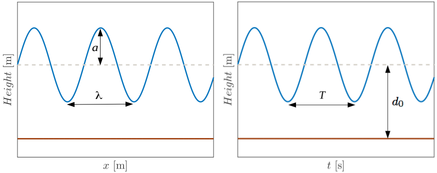
$$C^2 = C_0^2\,\frac{\tanh(\kappa\, d_0)}{\kappa\, d_0}$$

Phase relation and phase dispersion

$C_0^2 = g\,d_0$ shallow water (hyperbolic) celerity. $C = C(\kappa)$!!

Dispersive surface waves: Boussinesq approach (3/8)

a	wave amplitude
$\lambda = 2\pi/\kappa$	wave length
T	wave period
$\lambda = C(\kappa)T = C(2\pi/\lambda)T$	phase relation



Dimensionless parameters

- $\mu = \dfrac{d_0}{\lambda} = \dfrac{\kappa d_0}{2\pi}$ dispersion
- $\varepsilon = \dfrac{a}{d_0}$ nonlinearity

Physical hypotheses

- Long waves: smallness of μ^2
- Weak-nonlinearity: smallness of $\epsilon = \mathcal{O}(\mu^2)$
- Full-nonlinearity: $\epsilon = \mathcal{O}(1)$

Modelling principles[1]

Starting from the 3D nonlinear wave equations:

1. dimensionless form ;
2. asymptotic development w.r.t. μ^2: $\nabla\Phi = \nabla\Phi_0 + \mu^2\nabla\Phi_2 + \mu^4\nabla\Phi_4 + \ldots$
3. depth averaging: 3D $\rightarrow$ 2D
4. retain appropriate order terms ....

[1] J. Boussinesq, *J.Math.Pures Appl.*, 1872 – M.W. Dingemans, World Scientific, 1997

Dispersive surface waves: Boussinesq approach (5/8)

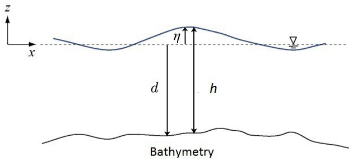
Example: shallow water equations (zero-th order model)

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = 0 \; \cancel{+\mathcal{O}(\mu^2)}$$

With the notation

- dimensionless depth : $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- dimensionless volume flux : $q = \tilde{h}\tilde{u}$
- dimensionless depth averaged velocity: $\tilde{u}$
- Red : hyperbolic shallow water equations
- Blue terms are responsible for dispersion



Nonlinear – non dispersive

Example: Peregrine's equations[2]

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \tilde{h}\left(\frac{\tilde{d}^2}{3} u_{txx} + \frac{\tilde{d}\tilde{d}_x}{3}\tilde{u}_{tx}\right) \cancel{+\mathcal{O}(\mu^4, \epsilon\mu^2)}$$

With the notation

- dimensionless depth : $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- dimensionless volume flux : $q = \tilde{h}\tilde{u}$
- dimensionless depth averaged velocity: $\tilde{u}$
- Red : hyperbolic shallow water equations
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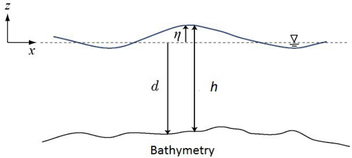

Weakly nonlinear – weakly dispersive

[2] D.H. Peregnine. J.Fluid.Mech, 1967

Dispersive surface waves: Boussinesq approach (7/8)

Example: Madsen & Sorensen's enhanced equations[3]

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \left(\beta \tilde{d}^2 \tilde{q}_{txx} + \frac{\tilde{d}\tilde{d}_x}{3} \tilde{q}_{tx} + B \tilde{d}^3 \tilde{\eta}_{xxx} + 2B \tilde{d}^2 \tilde{d}_x \tilde{\eta}_{xx} \right) \cancel{+ \mathcal{O}(\mu^4, \epsilon\mu^2)}$$

With the notation

- dimensionless depth : $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- dimensionless volume flux : $q = \tilde{h}\tilde{u}$
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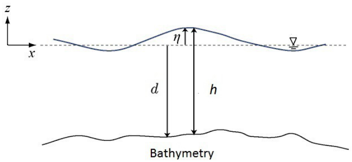


Weakly nonlinear – weakly dispersive

Phase enhancement vie the tunable coeff. B and $\beta = B + 1/3$ (cf. later)

[3] P.A. Madsen and O.R. Sorensen Coast.Eng., 1992

EXAMPLE: ENHANCED SERRE-GREEN-NAGHDI EQUATIONS[4]

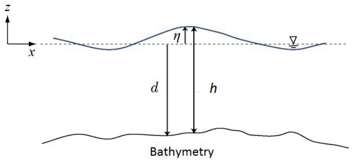
$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2\, \tilde{h}\tilde{\psi}\ \cancel{+\mathcal{O}(\mu^4)}$$

$$\tilde{\psi} = \alpha\left[\partial_x(\tilde{h}^2\partial_x(\tilde{u}_t + \tilde{u}\tilde{u}_x))\right] + (\alpha - 1)\left[\partial_x(\tilde{h}^2\partial_{xx}\tilde{\eta})\right] + \mathcal{Q}_\psi(\tilde{u}, \tilde{h}, \tilde{d}; \tilde{u}_x, \tilde{h}_x, \tilde{d}_x)$$

With the notation

- dimensionless depth : $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- dimensionless volume flux : $q = \tilde{h}\tilde{u}$
- dimensionless depth averaged velocity: $\tilde{u}$
- Red : hyperbolic shallow water equations
- Blue terms are responsible for dispersion



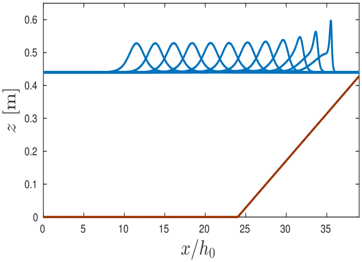
Fully nonlinear – weakly dispersive
Phase enhancement vie the tunable coeff. α (cf. later)

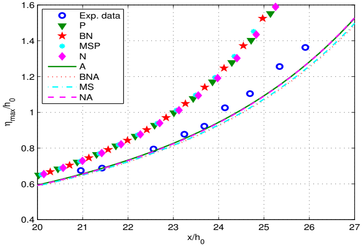
[4] A.E. Green J.Fluid Mech., 1976 – F. Chazel et al. J.Sci.Comp., 2011

Modelling error (1/3)

What are these models good for: nonlinear behavior

Shoaling test, weakly nonlinear models[5]





Many variations for a given asymptotic accuracy (same linear limit), e.g.

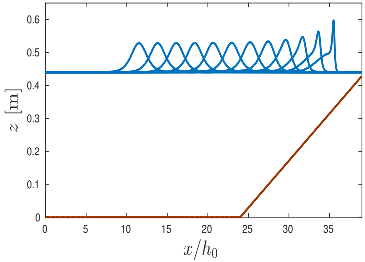
- $\mu^2 \tilde{d} \approx \mu^2 \tilde{h}$ as $\epsilon\mu^2\tilde{\eta}$ is negligible (weakly nonlinear)
- $\mu^2(\tilde{d}\tilde{u})_{xxt} \approx \mu^2\tilde{q}_{xxt}$ as $\epsilon\mu^2(\tilde{\eta}\tilde{u})_{xx}$ is negligible (weakly nonlinear)
- etc.

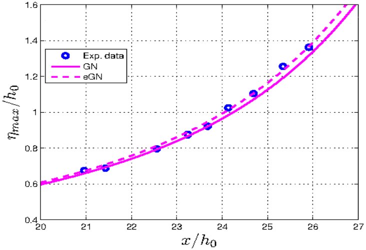
[5] S.T. Grilli et al J.Waterw.Port.C.-ASCE, 1994 – A.G. Filippini et al. Coast.Eng., 2015

Modelling error (2/3)

What are these models good for: nonlinear behavior

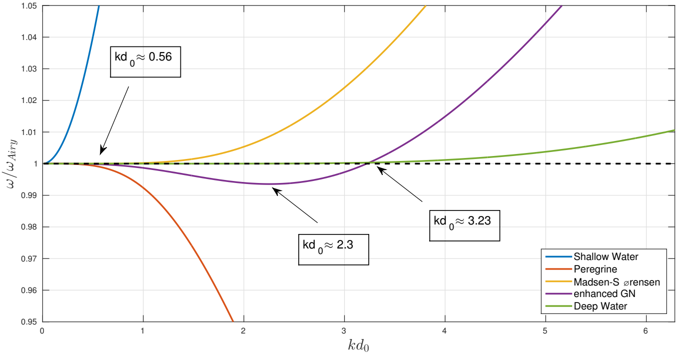
Shoaling test, fully nonlinear models





Modelling error (3/3)

What are these models good for: phase behavior



CAVEAT: MS and eGN are equivalent. The "classical" optimized MS obtained changing α

Continous and discrete

Continuous: error w.r.t. Euler eq.s

- Range of validity in terms of reduced wave number $\kappa d_0 = 2\pi d_0/\lambda$
- Near shore: $(\kappa d_0)_{\max} \approx \pi$, $(d_0/\lambda)_{\max} \approx 2$

Discrete: error w.r.t. continuous model

- Range of validity in terms of $1/N = \Delta x/\lambda$: pts per wavelength

Overall error w.r.t. Euler eq.s ???

EXAMPLE: LINEARIZED SHALLOW WATER

$$\partial_t W + A\partial_x W = 0$$

with (in dimensionless form)

$$W = \begin{pmatrix} \eta \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad |A| = \mathrm{I}d_2$$

where we recall that (all $\tilde{.}$ are removed for simplicity, and $c_0^2 = gd_0$)

$$\eta = \frac{\eta^{\text{dim}}}{\epsilon d_0}, \quad u = \frac{u^{\text{dim}}}{\epsilon c_0}, \quad x = \frac{x^{\text{dim}}}{\lambda}, \quad t = \frac{t^{\text{dim}}}{\lambda / c_0}$$

Truncation error heuristics

Example: linearized shallow water

1st order upwind method

$$\partial_t W + A \frac{W_{i+1} - W_{i-1}}{2\Delta x} = \frac{\Delta x}{2} \frac{W_{i+1} - 2W_i + W_{i-1}}{\Delta x^2}$$

2nd order centered differencing

$$\partial_t W + A \frac{W_{i+1} - W_{i-1}}{2\Delta x} = 0$$

Truncation error heuristics

Example: linearized shallow water

1st order upwind and 2nd order centered differencing: modified equation/TE

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = \frac{\Delta x}{2}\partial_{xx} W^{\text{smooth}} - \frac{A\Delta x^2}{6}\partial_{xxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^3)$$

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = - \frac{A\Delta x^2}{6}\partial_{xxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^4)$$

It looks like "*if we could see the dispersive effects of the 1st order scheme, they would be the same as those of the second order*"..

but the numerical viscosity is too high...

Example: linearized shallow water

Turn down the viscosity: 3rd upwind scheme

$$\partial_t W + \left(A \frac{W^R_{i+1/2} + W^L_{i+1/2}}{2\Delta x} - \frac{W^R_{i+1/2} - W^L_{i+1/2}}{2\Delta x} \right)$$
$$- \left(A \frac{W^R_{i-1/2} + W^L_{i-1/2}}{2\Delta x} - \frac{W^R_{i-1/2} - W^L_{i-1/2}}{2\Delta x} \right) = 0$$

With "*quadratically reconstructed*" left and right values at $x_i \pm \Delta x/2$

$$W^L_{i+1/2} = W_i + \frac{W_i - W_{i-1}}{6} + \frac{W_{i+1} - W_i}{3}$$
$$W^R_{i+1/2} = W_{i+1} - \frac{W_i - W_{i-1}}{6} - \frac{W_{i+2} - W_{i+1}}{3}$$

Example: linearized shallow water

3rd order upwind vs 4th order central differencing: modified eq./TE

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = -\frac{\Delta x^3}{12}\partial_{xxxx} W^{\text{smooth}} - \frac{A\Delta x^4}{30}\partial_{xxxxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^5)$$

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = -\frac{A\Delta x^4}{30}\partial_{xxxxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^6)$$

- Low dissipation: $\mathcal{O}(\Delta x^3)$ viscosity (ok for hyperbolic + explicit time stepping)
- Same dispersion of the fourth order FD !

Truncation error heuristics: dispersive models

Example: enhanced linearized Madsen-Sorensen

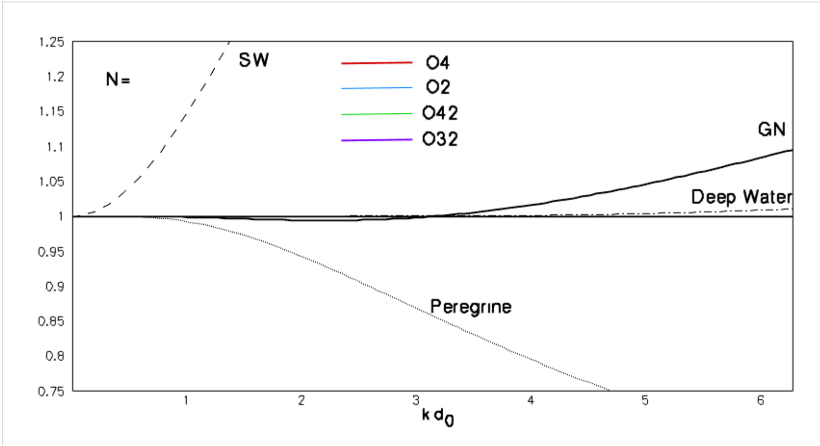
$$\begin{cases} \partial_t \eta + \partial_x u = 0 \\ \\ \partial_t u - \mu^2 (\dfrac{1}{3} + \beta) \partial_{xxt} u + \partial_x \eta - \mu^2 \beta \partial_{xxx} \eta = 0 \end{cases}$$

Example: enhanced linearized Madsen-Sorensen

$$
\begin{cases}
\partial_t \eta + \partial_x u = 0 \\
\partial_t w + \partial_x \zeta = 0 \\
u - \mu^2 (\frac{1}{3} + \beta) \partial_{xx} u = w \\
\eta - \mu^2 \beta \partial_{xx} \eta = \zeta
\end{cases}
$$

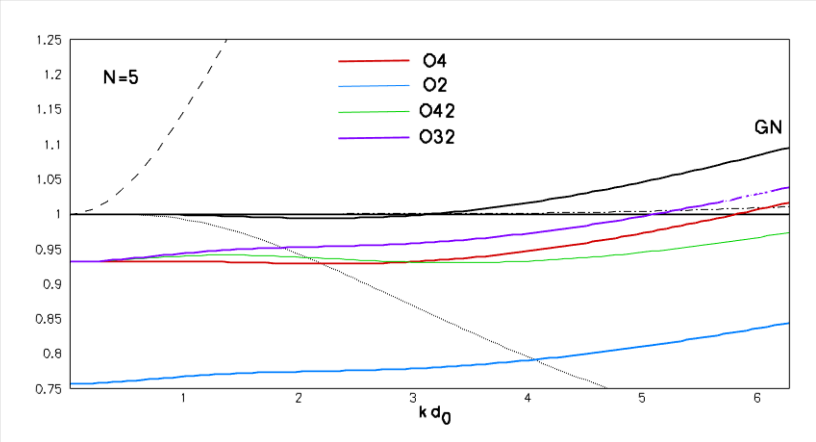
First order ("hyperbolic") system
Overhead w.r.t. hyperbolic system
How accurate must the discretization of these red terms be ?

Linearized Madsen-Sorensen (with GN α_{opt})



Dispersion error heuristics

Linearized Madsen-Sorensen (with GN α_{opt})



Dispersion error heuristics

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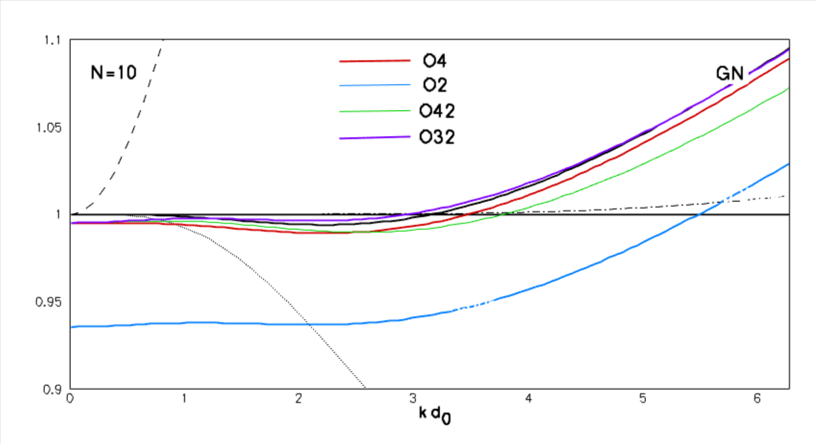
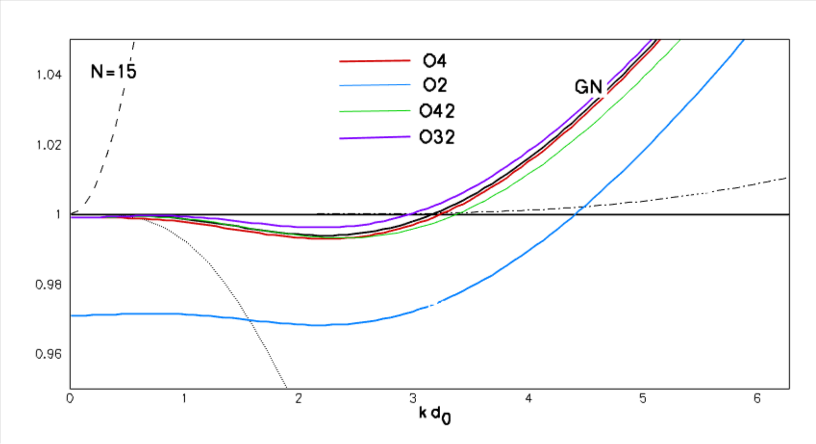
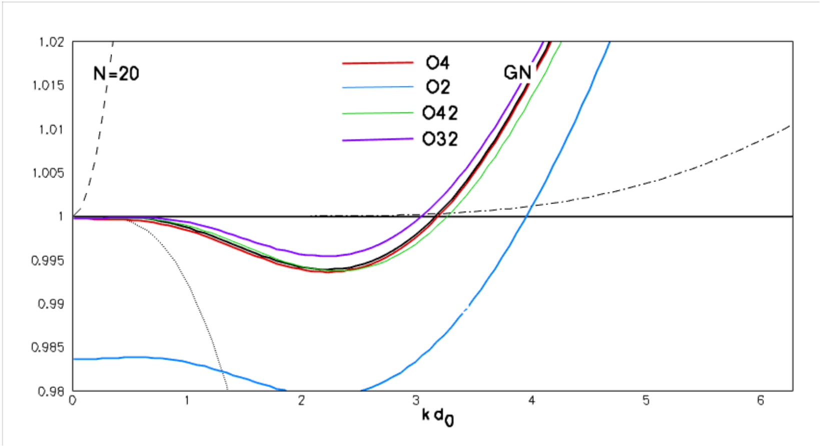


LINEARIZED MADSEN-SORENSEN (WITH GN α_{opt})



Dispersion error heuristics

Linearized Madsen-Sorensen (with GN α_{opt})



Lesson learned

Hyperbolic operator
We MUST use at least a third order scheme

Elliptic operator
Waste of effort using more than second order for near shore models

Next step
Unstructured grid generalization for the eGN model

THE eGN EQUATIONS (IN 1D)

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \, \tilde{h}\tilde{\psi}$$

$$\tilde{\psi} = \alpha \left[\partial_x (\tilde{h}^2 \partial_x (\tilde{u}_t + \tilde{u}\tilde{u}_x)) \right] + (\alpha - 1) \left[\partial_x (\tilde{h}^2 \partial_{xx} \tilde{\eta}) \right] + \mathcal{Q}_\psi(\tilde{u}, \tilde{h}, \tilde{d}; \tilde{u}_x, \tilde{h}_x, \tilde{d}_x)$$

With the notation

- dimensionless depth : $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- dimensionless volume flux : $q = \tilde{h}\tilde{u}$ with
- $\tilde{u}$ the (dimensionless) depth averaged velocity
- Red terms provide the hyperbolic shallow water equations
- Blue terms are responsible for dispersion

Fully nonlinear – weakly dispersive

Phase enhancement vie the tunable coeff. α (cf. later)

The eGN equations (in 2D)

$$\partial_t h + \nabla \cdot \mathbf{q} = 0\,;$$

$$\Big(\mathcal{I} + \alpha \mathrm{T}_h\Big)\Big(\partial_t \mathbf{q} + \nabla \cdot \Big(\frac{\mathbf{q} \otimes \mathbf{q}}{h}\Big) + gh\nabla\eta\Big) - \mathrm{T}_h\Big(gh\nabla\eta\Big) + h\mathcal{Q}(\mathbf{u}) = 0\,.$$

With the notation

- depth : $h = d + \eta$
- (vector) volume flux : $\mathbf{q} = \tilde{h}\mathbf{u}$ with
- $\mathbf{q}$ the (dimensionless) depth averaged velocity vector
- T_h the dispersive elliptic operator given by

$$\mathrm{T}_h(\cdot) = h\mathcal{T}(\frac{\cdot}{h})$$

where $\mathcal{T}$ is a self adjoint operator[6]:

$$\mathcal{T}(\cdot) = S^*(S(\cdot))\,, \qquad S(\cdot) = \frac{h}{\sqrt{3}}\nabla \cdot (\cdot) - \frac{\sqrt{3}-1}{2}\nabla b \cdot (\cdot)$$

[6] Alvarez-Samaniego and Lannes, Indiana Univ. Math J., 2008

The eGN equations

We recast the system in two independent steps:

$$\partial_t h + \nabla \cdot \mathbf{q} = 0\,;$$
$$\partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q}\otimes\mathbf{q}}{h}\right) + gh\nabla\eta = h\boldsymbol{\psi}\,; \longrightarrow \textbf{hyperbolic step}$$
$$\Big(\mathcal{I} + \alpha\mathcal{T}\Big)(\,\boldsymbol{\psi}\,) - \mathcal{T}(g\nabla\eta) + \mathcal{Q}(\mathbf{u}) = 0\,. \longrightarrow \textbf{elliptic step}$$

This reformulation aims at exploiting the self-adjoint character of $\mathcal{T}$

SOLUTION ALGORITHM

At each time-step n:

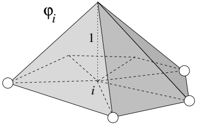
1. elliptic step is solved : $\psi = \big(\mathcal{I} + \alpha\mathcal{T}\big)^{-1}(\mathit{RHS})$ using $(h^n, \mathbf{q}^n)$;
2. shallow water solver + non-hydrostatic term $\psi \ \rightarrow \ (h^{n+1}, \mathbf{q}^{n+1})$.

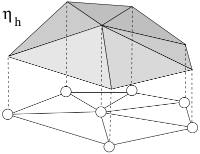
Elliptic step

- Continuous P^1 Galerkin FE exploiting the self-adjoint character of $\mathcal{T}$[7]
$$\int_\Omega \boldsymbol{\nu} \cdot \boldsymbol{\psi}_\mathsf{h} + \alpha \int_\Omega S(\boldsymbol{\nu})\, S(\boldsymbol{\psi}_\mathsf{h}) = \mathsf{RHS}(\eta_\mathsf{h}, h_\mathsf{h}, \boldsymbol{u}_\mathsf{h}, b_\mathsf{h}) \ ;$$

- Linear system :
$$(\mathbb{M}_\mathsf{H}^\mathsf{G} + \alpha \mathbb{T})\, \boldsymbol{\Psi} = \mathbb{T}\ \boldsymbol{\delta}_\mathsf{h} - \mathbb{Q} \quad \text{with } \boldsymbol{\delta}_\mathsf{h} \text{ the } L^2 \text{ projection of } g\nabla\eta$$

- Coercivity of $\mathcal{I} + \alpha\mathcal{T} \ \longrightarrow$ inversibility of $(\mathbb{M}_\mathsf{H}^\mathsf{G} + \alpha\mathbb{T})$
(block SPD + diagonally dominant) ;



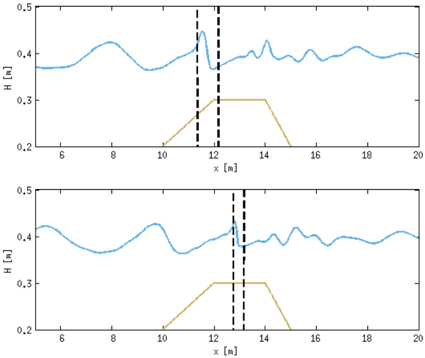


[7] Alvarez-Samaniego and Lannes Indiana Univ. Math J., 2008

Discretization of the (breaking) eGN on unstructured grids

Elliptic step

- Propagation: add $h\psi$ to the rhs
- Wave breaking:
  1. Flag nodes
  2. Agglomerate elements and enlarge breaking region in wave direction
  3. Set ψ to zero: breakers as shallow water shocks[9]
- Wave breaking detection[10]
  - either $|\eta_t| > \gamma\sqrt{gH}$ with $\gamma \in [0.4,\, 0.6]$
  - or $\|\nabla\eta\| > \text{tg}\theta$ with $\theta \in [15,\, 30]^\circ$
  - and $\text{Fr} > \text{Fr}_{cr}$

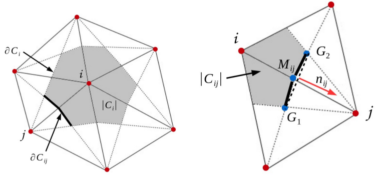


[9] P. Bonneton Ocean Eng., 2007

[10] Kazolea, Delis and Synolakis, JCP 2014

Discretization of the (breaking) eGN on unstructured grids

Hyperbolic step



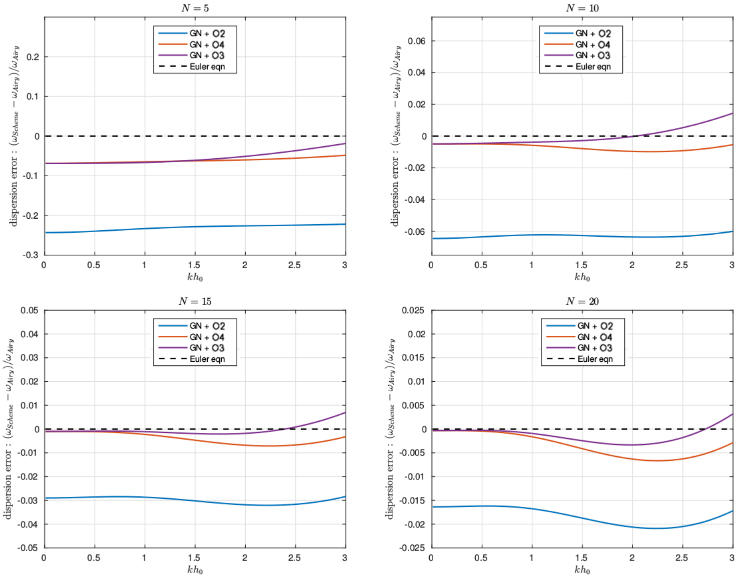
- Node - centered FV scheme
- $$\frac{\partial \mathbf{U}_i}{\partial t} + \frac{1}{|C_i|} \int_{\partial\Omega} \left(\mathbf{F}\hat{n}^x + \mathbf{G}\hat{n}^y \right) = \frac{1}{|C_i|} \int_{\Omega} \left(\mathbf{S}_b + h\mathbf{\Psi} \right) \; ;$$
- Roe's Riemann solver + Harten-Hyman entropy fix
- High order reconstruction: weighted least squares[11] (quadratic or cubic) ;
- Well-balanced treatment of topography, wet/dry fronts, etc[12]

[11] Ollivier-Gooch et al. AIAA J. 2009; Wang et al JCP 2017

[10] Bermudez&Vazquez, CAF 1994; Hubbard&Garcia-Navarro, JCP 2000; Brufau et al. IJNMF 2002; Castro, Math.&Computer Mod. 2005; etc.etc.

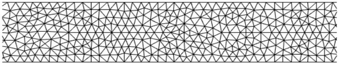
Discretization of the (breaking) eGN on unstructured grids

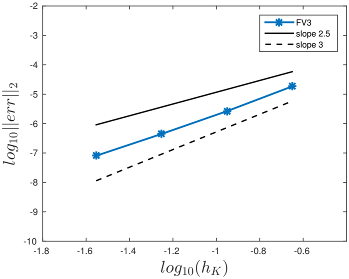
Dispersion analysis of the scheme

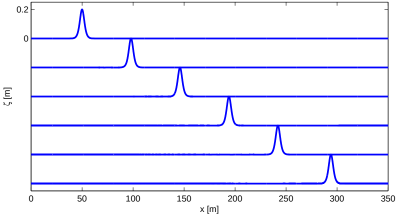


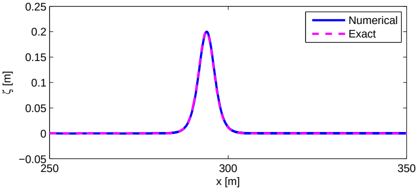
Benchmarking: propagation test 1

Test Description: $a_0 = 0.2$ [m], $h_0 = 1$ [m]



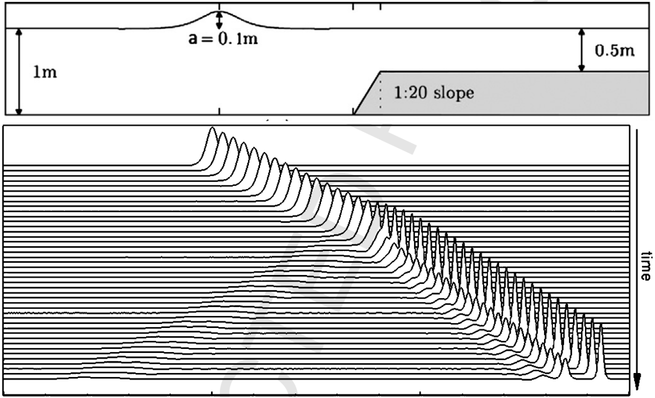






Benchmarking: propagation test 2

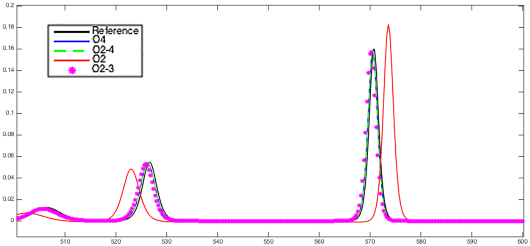
Shoaling on a shelf



Benchmarking: propagation test 2

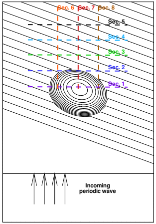
Shoaling on a shelf

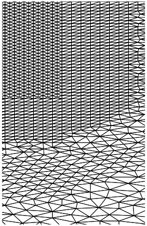
Final solution

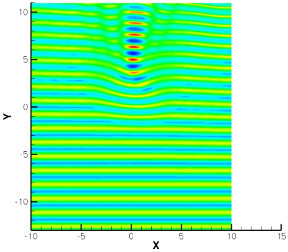


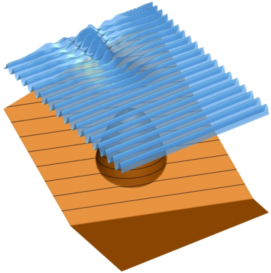
Benchmarking: diffraction on an elliptic shoal

- Range : $a/h_0 = 0.0515$, $T = 1$ [s] ;
- Energy transfer to higher harmonics;
- Experiments: (*Berkhoff et al.*, 1982)
- Adapted mesh 88760 nodes ;



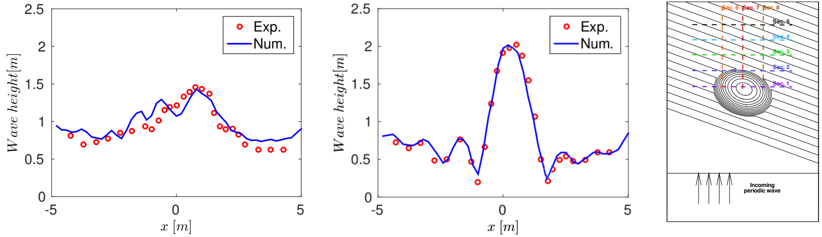




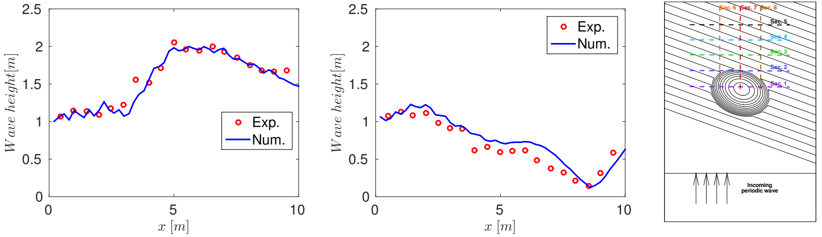


Benchmarking: diffraction on an elliptic shoal

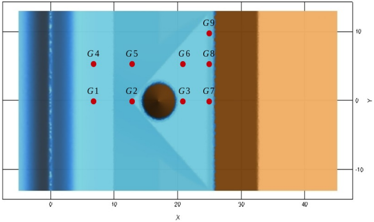
Section 2 and Section 4

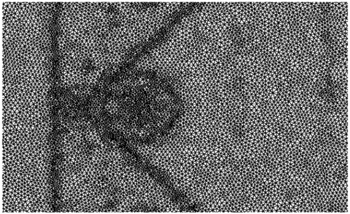


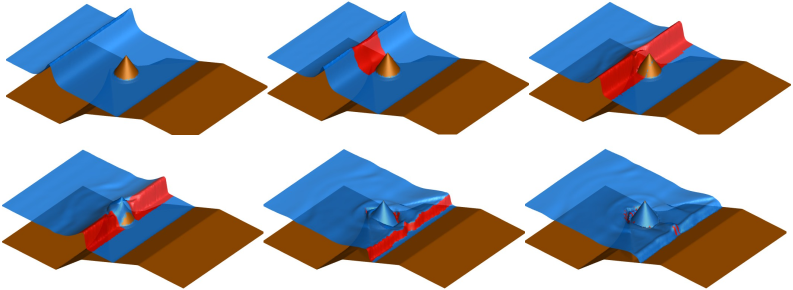
Section 7 and Section 8



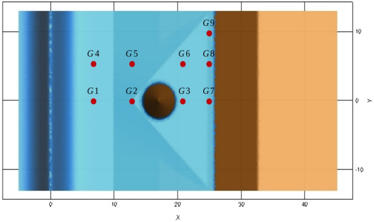
Benchmarking: overtopping on a three dimensional reef

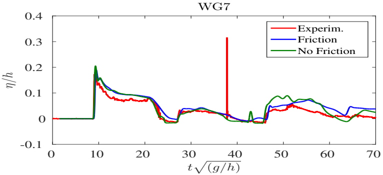


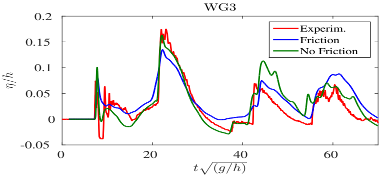


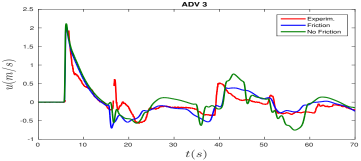


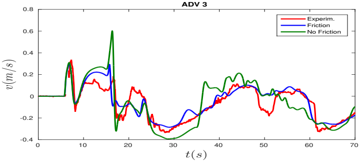
Benchmarking: overtopping on a three dimensional reef







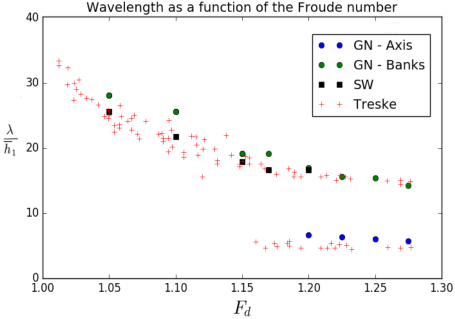




Application: study of undular bores and Treske experiments

Undular bore (Garonne river,
Bonneton et al *J.Geophys.Res.* 2015)

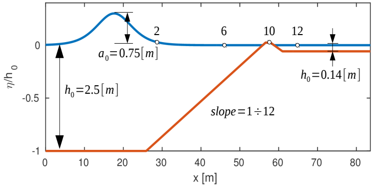
Application: study of undular bores and Treske experiments



Comparison with experiments by Treske

The short of it ...

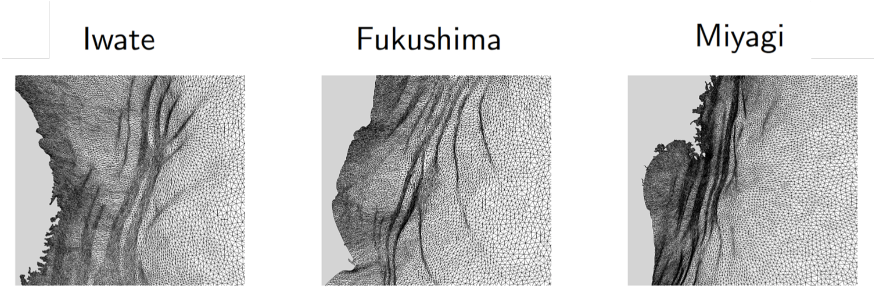
- Interaction modelling/discretization error for near shore Boussinesq
- Frst (hyperbolic) order system vs second order elliptic operator :
  - High order on hyperbolic : third order (at least) for good dispersion
  - Elliptic component: can be treated with a second order method
  - Unstructured grid eGN: high(er) order FV + P1 FEM for elliptic part
- Wave breaking: revert to SW + shock capturing



Perspectives and open issues

A few ongoing/foreseen extensions

- Systematic orders/CPU time investigation
- Other methods, in particular DG and RD
- Implicit time integration + energy conserving in space and time ?
- Wave breaking via PDE based eddy viscosity
- Study of deep water/non-hydrostatic (multi-layer) models
- Moving meshes and adaptation
  (cf. parallel session on rupture based tsunami simulation)



Tohoku Tsunami, March 11, 2011